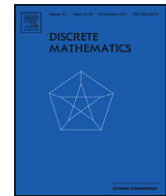


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## Discrete Mathematics

journal homepage: [www.elsevier.com/locate/disc](http://www.elsevier.com/locate/disc)On  $K_{s,t}$ -minors in graphs with given average degree, IIA.V. Kostochka<sup>a,b,\*</sup>, N. Prince<sup>c</sup><sup>a</sup> Department of Mathematics, University of Illinois, Urbana, IL, 61801, USA<sup>b</sup> Sobolev Institute of Mathematics, Novosibirsk, Russia<sup>c</sup> Illinois Mathematics and Science Academy, Aurora, IL, 60506, USA

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## ABSTRACT

Let  $K_{s,t}^*$  denote the graph obtained from  $K_{s,t}$  by adding all edges between the  $s$  vertices of degree  $t$  in it. We show how to adapt the argument of our previous paper [A.V. Kostochka, N. Prince, On  $K_{s,t}$ -minors in graphs with given average degree, Discrete Math. 308 (2008) 4435–4445] to prove that if  $t/\log_2 t \geq 1000s$ , then every graph  $G$  with average degree at least  $t + 8s \log_2 s$  has a  $K_{s,t}^*$  minor. This refines a corresponding result by Kühn and Osthus.

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## 1. Introduction

Graphs in this paper are undirected simple graphs. For a graph  $G$ ,  $V(G)$  is the set of its vertices,  $E(G)$  is the set of its edges,  $e(G) = |E(G)|$ , and  $v(G) = |V(G)|$ . By  $G[X]$  we denote the subgraph of  $G$  induced by the vertex set  $X$ . We let  $N_G(v)$  denote the set of neighbors of  $v$  in  $G$  and  $N_G[v] = N_G(v) \cup \{v\}$ . Similarly, for  $X \subseteq V(G)$ , we define  $N(X) := \bigcup_{x \in X} N(x)$ . A *minor* of a graph  $G$  is a graph  $H$  that can be obtained from  $G$  by a sequence of vertex and edge deletions and edge contractions. For a graph  $H$ , let  $D(H)$  denote the minimum number  $t$  such that every graph  $G$  with average degree at least  $t$  has an  $H$ -minor, i.e., a minor isomorphic to  $H$ .

Mader [8] proved that  $D(K_r) \leq 8r \ln r$ . Later, Kostochka [2,3] and Thomason [13] showed that the order of magnitude of  $D(K_r)$  is  $r\sqrt{\ln r}$ , and then Thomason [14] found the asymptotics of  $D(K_r)$  as  $r \rightarrow \infty$ . Myers and Thomason [12,9] determined  $D(H)$  for almost every  $H$ , showing, in particular, that for almost all  $H$ , the extremal graphs not containing  $H$  are quasi-random (built deterministically from randomly generated subcomponents). Their methods work better for dense and balanced graphs  $H$ .

An example of a sparse and unbalanced  $H$  is the complete bipartite graph  $K_{s,t}$ , where  $s$  is fixed and  $t$  is large with respect to  $s$ . For this reason, Myers [10,11] studied  $D(K_{s,t})$  when  $s$  is fixed and  $t$  is large. Let  $M(r, s, t)$  be the graph obtained by taking  $r$  copies of  $K_{s+t-1}$  arranged so that each two copies share the same fixed  $s-1$  vertices (Fig. 1 shows  $M(2, 3, 4)$ ). Myers [11] observed that  $M(r, s, t)$  has no  $K_{s,t}$ -minor and that

$$e(M(r, s, t)) = \frac{1}{2}(t + 2s - 3)(n - s + 1) + \binom{s-1}{2}, \quad (1)$$

where  $n = |V(M(r, s, t))| = rt + s - 1$ . He proved that for  $t > 10^{29}$  and  $n \geq 3$ , each  $n$ -vertex graph  $G$  with more than  $\frac{1}{2}(t+1)(n-1)$  edges has a  $K_{2,t}$  minor. The graphs  $M(r, 2, t)$  witness that this bound is sharp when  $|V(G)| \equiv 1 \pmod{t}$ . In connection with graph coloring, Chudnovsky et al. [1] proved that Myers' bound is true for all  $t$ .

Myers conjectured that a similar, more general statement holds for  $K_{s,t}$ -minors.

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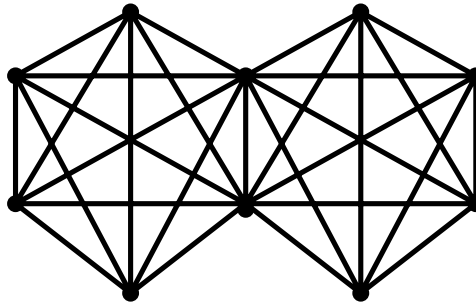


Fig. 1. Graph  $M(2, 3, 4)$  has no  $K_{3,4}$ -minor.

**Conjecture 1.** Let  $s$  be a positive integer. Then there exists a constant  $C(s)$  such that, for all positive integers  $t$ , if  $G$  has average degree at least  $C(s) \cdot t$ , then  $G$  has a  $K_{s,t}$ -minor.

Let  $K_{s,t}^* = K_{s+t} - E(K_t)$ . In other words,  $K_{s,t}^*$  is the graph obtained from  $K_{s,t}$  by adding all  $\binom{s}{2}$  possible edges into the  $s$ -vertex partite set. Myers noted that the average degree that forces  $G$  to contain a  $K_{s,t}$ -minor also likely forces a  $K_{s,t}^*$ -minor, that is,  $D(K_{s,t}) = D(K_{s,t}^*)$  when  $s$  is fixed and  $t$  is large.

Myers' Conjecture was proved for large  $t$  independently in [5,7] using different methods. Kühn and Osthus [7] showed the following.

**Theorem 1** ([7]). For every  $0 < \epsilon < 10^{-16}$ , there exists a number  $t_0 = t_0(\epsilon)$  such that for all integers  $t \geq t_0$  and  $s \leq \epsilon^6 t / \log t$ , every graph of average degree at least  $(1 + \epsilon)t$  contains  $K_{s,t}$  as a minor.

They also showed that  $K_{s,t}$  can be replaced with  $K_{s,t}^*$  if the restriction  $s \leq \epsilon^6 t / \log t$  is replaced with  $s \leq \epsilon^7 t / \log t$ .

In [5], the following fact was proved.

**Theorem 2.** Let  $n, s$  and  $t$  be positive integers with

$$t > (240s \log_2 s)^{8s \log_2 s + 1}.$$

Let  $G$  be an  $n$ -vertex graph such that  $e(G) \geq \frac{t+3s}{2}(n-s+1)$ . Then  $G$  has a  $K_{s,t}^*$ -minor. Furthermore, for infinitely many  $n$ , there exists a graph  $G_n$  of order  $n$  and size at least  $\frac{t+3s-5\sqrt{s}}{2}(n-s+1)$  that has no  $K_{s,t}$ -minor.

From Theorem 2 we have that for huge  $t$ ,

$$t + 3s - 5\sqrt{s} \leq D(K_{s,t}) \leq D(K_{s,t}^*) \leq t + 3s.$$

Hence, Myers' insight that  $D(K_{s,t})$  is the same as  $D(K_{s,t}^*)$  is true asymptotically in  $s$ .

Observe that while Theorem 1 provides a weaker bound on the second term of  $D(K_{s,t})$  (essentially, the second term in their bound is  $(s \ln t / t)^{1/6} t$  while in Theorem 2 it is the asymptotically (in  $s$ ) exact  $3s$ ), it applies for a much wider (essentially best possible) range of  $t$  for a given  $s$  than Theorem 2, namely for  $t \geq C \cdot s \log t$ . Kühn and Osthus [7] also proved the following fact showing that the statement of their theorem would be incorrect if  $s \geq 18t / \ln t$ .

**Proposition 3** ([7, Proposition 10]). There exists  $n_0$  such that for each integer  $n \geq n_0$  and each  $\alpha > 0$ , there is an  $n$ -vertex graph  $G$  with average degree at least  $n/2$  that does not have a  $K_{s,t}$  minor with  $s = \lceil 2n/\alpha \ln n \rceil$  and  $t = \lceil \alpha n \rceil$ .

In particular, it implies that the statement of Theorem 2 is not correct when  $t = s \ln s$  and  $s$  is large.

The goal of the present note is to show how to adapt the proof of Theorem 2 to prove the following.

**Theorem 4.** Let  $1 \leq s \leq t \leq n$  be integers such that  $n \geq 2s$  and

$$s \leq t/1000 \log_2 t. \quad (2)$$

Let  $G$  be an  $n$ -vertex graph with  $e(G) \geq \frac{t+8s \log_2 s}{2}(n-s+1)$ . Then  $G$  has a  $K_{s,t}^*$ -minor.

This theorem applies to the same range of  $t$  in terms of  $s$  as Theorem 1 (and even a bit better, since the range does not depend on  $\epsilon$ ), but gives the better estimate of the second term. The first author plans to use Theorem 4 to improve the result of [4], where Theorem 2 was used.

The idea of this note is that in the proof of Theorem 2, we needed  $t$  that is much larger than  $s$  only in the case when  $n$  is small, essentially when  $n < t + C \cdot s \ln t$ . This note shows that in this range we can prove the bound of Theorem 4 (which is weaker) for  $n \leq t + C \cdot s \ln s$ . The setup and this case are handled in Section 2. In Section 3 we list useful lemmas from [5] and in Section 4 we present the proof of the main case.

## 2. Setup and graphs of small order

In [6] we proved the following.

**Theorem 5.** Let  $t \geq 6300$ . Let  $G$  be a graph of order  $n \geq 3$  with  $e(G) > \frac{1}{2}(t+3)(n-2) + 1$ . Then  $G$  has a  $K_{3,t}^*$ -minor.

So, it is enough to prove Theorem 4 for  $s \geq 4$ . Similarly to the proof in [5], we say that a graph  $G$  is  $(s, t)$ -irreducible if

- (i)  $v(G) \geq 2s$ ;
- (ii)  $e(G) \geq 0.5(t + 8s \log_2 s)(v(G) - s + 1)$ ;
- (iii)  $G$  has no proper minor  $G'$  possessing (i) and (ii).

For an edge  $e$  of a graph  $G$ ,  $t_G(e)$  denotes the number of triangles in  $G$  containing  $e$ . Similarly to Lemma 3 in [5], the following lemma holds.

**Lemma 6.** If  $G$  is an  $n$ -vertex  $(s, t)$ -irreducible graph and  $s < t/1000 \log_2 t$ , then

- (a)  $n \geq t + 8s \log_2 s - 1.5s + 1$ ;
- (b)  $t_G(e) \geq 0.5(t + 8s \log_2 s) - 1$  for every  $e \in E(G)$ ;
- (c) if  $W \subset V(G)$  and  $v(G) - |W| \geq 2s$ , then  $W$  is incident with at least  $0.5(t + 8s \log_2 s)|W|$  edges; in particular,  $\delta(G) \geq 0.5(t + 8s \log_2 s)$ ;
- (d)  $G$  is  $s$ -connected;
- (e)  $e(G) < 0.5(t + 8s \log_2 s)n$ .

**Proof.** The proofs of (b)–(d) are almost exactly the same as in the proof of Lemma 3 in [5]. Property (e) follows from the fact by (iii), deleting any edge from  $G$  results in a graph not possessing (ii). So we present only the proof of (a), which slightly differs from that of Lemma 3 in [5]. Since  $G$  has at most  $\binom{n}{2}$  edges, for the quadratic function  $f(n) := n^2 - n - (t + 8s \log_2 s)(n - s + 1)$  we have  $f(n) \geq 0$ . The roots of  $f(n)$  are

$$n_{1,2} = \frac{1}{2} \left( t + 8s \log_2 s + 1 \pm \sqrt{(t + 8s \log_2 s + 1)^2 - 4(t + 8s \log_2 s)(s - 1)} \right).$$

Since  $(t + 8s \log_2 s + 1)^2 - 4(t + 8s \log_2 s)(s - 1) > (t + 8s \log_2 s - 3s + 1)^2$  for  $t \geq 1000s \log_2 t$ , either  $n < 1.5s$  or  $n > t + 8s \log_2 s - 1.5s + 1$ . This together with (i) proves (a).  $\square$

Since we aim for a weaker bound than in [5], instead of Lemmas 4–6 there, we prove just one.

**Lemma 7.** Each  $(s, t)$ -irreducible graph with no  $K_{s,t}^*$ -minor has at least  $10t/9$  vertices.

**Proof.** Suppose that an  $(s, t)$ -irreducible graph  $G$  has  $n = t + d$  vertices, where  $d \leq t/9$ . By Lemma 6(a),  $d \geq 8s \log_2 s - 1.5s + 1$ . If at most  $s - 1$  vertices of  $G$  have degree greater than  $t$ , then  $2e(G) \leq (s - 1)n + t(n - s + 1) = tn + d(s - 1) \leq tn + (s - 1)t/9$ . Since  $n > t > 1000s \log t$ , this is less than  $(t + 8s \log_2 s)(n - s + 1)$ , a contradiction to (ii). So, we may assume that vertices  $v_1, \dots, v_s$  have degree at least  $t$  in  $G$ .

Let  $k = \lceil \log_{3/2} d \rceil$ . We will find  $s$  disjoint dominating sets  $S_i$  with  $|S_i| \leq k + 1$  for each  $1 \leq i \leq s$ .

Initialize  $S_i^0 = \{v_i\}$  for  $1 \leq i \leq s$ . For  $1 \leq i \leq s$ , consecutively run the following procedure. Define  $U_i^j$  be the set of vertices not dominated by  $S_i^{j-1}$ . By the choice of  $v_1, \dots, v_s$ ,  $|U_i^1| \leq d - 1$  for  $1 \leq i \leq s$ . Step  $j$  for constructing  $S_i$  is as follows. If  $U_i^j$  is empty or  $j = k + 1$ , then set  $S_i = S_i^j$  and stop. Otherwise, define  $W_i^j = V(G) - U_i^j - \bigcup_{q=1}^{j-1} S_q - S_i^{j-1} - \{v_{i+1}, \dots, v_s\}$  and let  $S_i^{j+1}$  be obtained from  $S_i^j$  by adding a vertex  $v_i^j \in W_i^j$  that has the most neighbors in  $U_i^j$ .

Since for every  $i$ , we do at most  $k$  steps and in each Step  $j$  add at most one vertex to  $S_i^j$ , we have  $|S_i| \leq k + 1$  for every  $i$ . It follows that for all  $i$  and  $j$ ,

$$|V(G) - W_i^j| = \left| U_i^j \cup S_i^{j-1} \cup \{v_{i+1}, \dots, v_s\} \cup \bigcup_{q=1}^{i-1} S_q \right| \leq |U_i^1| + s(k + 1) < d + s(k + 1). \quad (3)$$

By Lemma 6(c),  $\delta(G) \geq 0.5(t + 8s \log_2 s)$ . So by (3) for all  $i$  and  $j$  and each  $u \in U_i^j$ ,

$$\frac{|N_G(u) \cap W_i^j|}{|W_i^j|} \geq \frac{\delta(G) - |V(G) - W_i^j|}{n - |V(G) - W_i^j|} \geq \frac{0.5t - (d + s(k + 1))}{n - (d + s(k + 1))} \geq \frac{\frac{7t}{18} - s(k + 1)}{t - s(k + 1)}.$$

Since  $k + 1 \leq 2 + \log_{3/2} d \leq \log_{3/2} \left(\frac{3}{2}\right)^2 \frac{t}{9} \leq 1.71 \log_2 \frac{t}{4}$ , we conclude from (2) that for all  $i$  and  $j$  and each  $u \in U_i^j$ ,

$$\frac{|N_G(u) \cap W_i^j|}{|W_i^j|} \geq \frac{7}{18} - \frac{s(k + 1)}{t} \geq \frac{7}{18} - \frac{s(1.71 \log_2 t)}{t} > \frac{1}{3}. \quad (4)$$

Hence by the choice of  $v_i^j$ , it has at least  $\frac{|U_i^j|}{3}$  neighbors in  $U_i^j$ . Thus for all  $i$  and  $j$ ,  $|U_i^j| < (2/3)^j d$ . In particular,

$$|U_i^{k+1}| < d \left( \frac{2}{3} \right)^{\lceil \log_{3/2} d \rceil} \leq d \cdot \frac{1}{d} = 1.$$

It follows that  $U_i^{k+1}$  is empty for  $1 \leq i \leq s$ . This means  $S_i$  is a dominating set of  $G$  with size at most  $k+1$ . Also,  $G[S_i]$  is connected because each  $v_i^j$  was chosen among the neighbors of  $S_i^{j-1}$ . Since  $s \geq 4$  and  $d \geq 8s \log_2 s - 1.5s + 1 \geq 59$ ,

$$\frac{s(k+1)}{d} \leq \frac{s(1.71 \log_2 (\frac{9}{4}d))}{d} \leq \frac{1.71s \log_2 (18s \log_2 s)}{7s \log_2 s} \leq \frac{1.71(\log_2 s + \log_2 \log_2 s + 4.18)}{7 \log_2 s}. \quad (5)$$

The derivative of the function  $\phi(x) = \frac{x + \log_2 x + 4.18}{x}$  is negative for  $x \geq 2$ . It follows that for  $s \geq 4$  by (5) we have

$$\frac{s(k+1)}{d} \leq \frac{1.71(\log_2 4 + \log_2 \log_2 4 + 4.18)}{7 \log_2 4} = \frac{1.71 \cdot 7.18}{14} < 1.$$

So, after contracting each  $S_i$  to a single vertex, there are at least  $n - d = t$  vertices that we did not touch. Thus the resulting graph contains  $K_{s,t}^*$ .  $\square$

### 3. Lemmas

The statements and proofs of Lemmas 7–10 in [5] do not need any change, since no relation between  $s$  and  $t$  is involved there. We will refer to the following two of them.

**Lemma 8** ([5, Lemma 9]). Let  $s, k$ , and  $n$  be positive integers and  $\alpha \geq 2$ . Suppose that  $n \leq \alpha(k+1)$ . Let  $G$  be a  $(3s \log_{\alpha/(\alpha-1)} n)$ -connected graph with  $n$  vertices and  $\delta(G) \geq k + 3(s-1) \log_{\alpha/(\alpha-1)} n$ . Then  $V(G)$  contains  $s$  disjoint subsets  $A_1, \dots, A_s$  such that for every  $i = 1, \dots, s$ ,

- (i)  $G[A_i]$  is connected;
- (ii)  $|A_i| \leq 3 \log_{\alpha/(\alpha-1)} n$ ;
- (iii)  $A_i$  dominates  $G - A_1 - \dots - A_{i-1}$ .

**Lemma 9** ([5, Lemma 10]). Let  $H$  be a graph and  $k$  be a positive integer. If  $C$  is an inclusion minimal  $k$ -separable set in  $H$  and  $S = N(C) - C$ , then the subgraph of  $H$  induced by  $C \cup S$  is  $(1 + \lceil \frac{k}{2} \rceil)$ -connected.

The statement of Lemma 11 in [5] also is correct in our setting, and the proof smoothly goes through when  $s \geq 4$  and  $t / \log_2 t > 1000s$ . It will be our main tool:

**Lemma 10** ([5, Lemma 11]). Let  $G$  be a  $100s \log_2 t$ -connected graph. Suppose that  $G$  contains a vertex subset  $U$  with  $t + 100s \log_2 t \leq |U| \leq 3t$  such that  $\delta(G[U]) \geq 0.4t + 100s \log_2 t$ . Then  $G$  has a  $K_{s,t}^*$ -minor.

### 4. Handling large graphs

The proof in the last section of [5] also works with small changes (we need some changes, since the range of  $t$  is different), but for convenience of the reader, instead of pointing out and commenting the differences we present below an updated version of this proof.

If Theorem 4 does not hold, then there exists an  $(s, t)$ -irreducible graph  $G$  with no  $K_{s,t}^*$ -minor. Let  $n = v(G)$ . By Lemma 7,  $n \geq 10t/9$ .

Case 1.  $G$  is  $200s \log_2 t$ -connected. If  $G$  has a vertex  $v$  with  $t + 100s \log_2 t \leq \deg(v) \leq 3t - 1$ , then  $G$  satisfies Lemma 10 with  $U = N[v]$  and we are done. Thus, we can assume that every vertex in  $G$  has either ‘small’ ( $< t + 100s \log_2 t$ ) or ‘large’ ( $\geq 3t$ ) degree. Let  $V_0$  be the set of vertices of ‘small’ degree. If  $|V_0| > t + 100s \log_2 t$ , then there is some  $V'_0 \subseteq V_0$  such that

$$t + 100s \log_2 t \leq \left| \bigcup_{v \in V'_0} N[v] \right| \leq 3t - 1.$$

In this case, we can apply Lemma 10 with  $U = \bigcup_{v \in V'_0} N[v]$ .

Now, let  $|V_0| \leq t + 100s \log_2 t$ . By Lemma 6(e), the average degree of  $G$  is less than  $t + 8s \log_2 s$ . Since every vertex outside of  $V_0$  has degree at least  $3t$ , we get

$$0.5t|V_0| + 3t(n - |V_0|) < (t + 8s \log_2 s)n$$

and hence by (2),  $n < \frac{2.5|V_0|}{2 - 8s \log_2 s/t} < 3t$ . Since (again by (2))  $n \geq 10t/9 > t + 100s \log_2 t$ , we can apply Lemma 10 with  $U = V(G)$  to find a needed minor.

Case 2.  $G$  is not  $200s \log_2 t$ -connected. Let  $S$  be a separating set with at most  $k = \lceil 200s \log_2 t \rceil - 1$  vertices and let  $V(G) - S = V_1 \cup V_2$  where vertices in  $V_1$  are not adjacent to vertices in  $V_2$ . Then each of  $V_1$  and  $V_2$  is a  $k$ -separable set. For  $j = 1, 2$ , let  $W_j$  be an inclusion minimal  $k$ -separable set contained in  $V_j$  and  $S_j = N(W_j) - W_j$ . By Lemma 9, the graph  $G_j = G[W_j \cup S_j]$  is  $100s \log_2 t$ -connected.

Case 2.1.  $|W_j \cup S_j| \geq t + 100s \log_2 t$  for some  $j \in \{1, 2\}$ . Then  $|W_j| \geq t - 100s \log_2 t$ . Let  $G_j = G[W_j \cup S_j]$ . By Lemma 6(b),  $\delta(G_j) \geq 0.5(t + 8s \log_2 s)$ . If  $|W_j \cup S_j| \leq 3t$ , then we apply Lemma 10 with  $U = W_j \cup S_j$ . So suppose

$$|W_j \cup S_j| > 3t. \quad (6)$$

As in Case 1, we may suppose that the degree of each  $w \in W_j$  is either 'small' ( $< t + 100s \log_2 t$ ) or 'large' ( $\geq 3t$ ). Let  $W'_j$  be the set of vertices  $w \in W_j$  of 'small' degree. As in Case 1, we conclude that  $|W'_j| \leq t + 100s \log_2 t$ . Since every vertex in  $W_j - W'_j$  has degree at least  $3t$ , we get

$$0.5t|W'_j| + 3t|W_j - W'_j| < (t + 8s \log_2 s)|W_j \cup S_j|.$$

Since  $|S_j| \leq k$ , by (6),  $3t|W_j - W'_j| \geq 3t(|W_j \cup S_j| - k - |W'_j|) \leq (3t - k)|W_j \cup S_j| - 3t|W'_j|$ . So again by (2),

$$|W_j \cup S_j| \leq \frac{2.5|W'_j|}{2 - (k + 8s \log_2 s)/t} \leq \frac{2.5(1.1t)}{2 - 0.208} < 3t,$$

a contradiction to (6).

Case 2.2.  $|W_j \cup S_j| < t + 100s \log_2 t$  for both  $j \in \{1, 2\}$ . Let  $H_j = G[W_j]$ . By Lemma 6(c) and the fact that  $|S_j| \leq k$ ,

$$\delta(H_j) \geq 0.5t - |S_j| \geq 0.5t - k. \quad (7)$$

Suppose that  $S_0$  is a separating set in  $H_j$  with  $|S_0| < 100s \log_2 t$ . Let  $W_j - S = W_{j,1} \cup W_{j,2}$  where vertices in  $W_{j,1}$  are not adjacent to vertices in  $W_{j,2}$ . For  $\ell = 1, 2$ , let  $e'(W_{j,\ell})$  denote the number of edges incident to  $W_{j,\ell}$ . By Lemma 6(c),  $e'(W_{j,\ell}) \geq 0.5(t + 8s \log_2 s)|W_{j,\ell}|$ . Since  $e_G(S_j \cup S_0, W_{j,\ell}) \leq |S_j \cup S_0| |W_{j,\ell}|$ , we have

$$\sum_{w \in W_{j,\ell}} d_G(w) = 2e'(W_{j,\ell}) - e_G(S_j \cup S_0, W_j) > (t - 1.5k)|W_j|.$$

It follows that some  $w_\ell \in W_{j,\ell}$  has degree greater than  $t - 1.5k$ . Thus,

$$\begin{aligned} 2(t - 1.5k) &\leq d_G(w_1) + d_G(w_2) < (|W_{j,1}| + |S_j \cup S_0|) + (|W_{j,2}| + |S_j \cup S_0|) \\ &\leq |W_j \cup S_j| + |S_j| + |S_0| \leq (t + 100s \log_2 t) + 100s \log_2 t + k. \end{aligned}$$

So,  $t < 200s \log_2 t + 4k < 1000s \log_2 t$ , a contradiction to (2). Therefore,  $H_j$  is  $100s \log_2 t$ -connected. By this, (7), and Lemma 8 (for  $k = 0.3t$  and  $\alpha = 4$ ),  $V(H_j)$  contains  $s$  disjoint subsets  $A_1^j, \dots, A_s^j$  such that for every  $i = 1, \dots, s$ ,

- (i)  $G[A_i^j]$  is connected;
- (ii)  $|A_i^j| \leq 3 \log_{4/3} |W_j| < 7.23 \log_2 |W_j| \leq 7.23 \log_2 (1.1t)$ ;
- (iii)  $A_i^j$  dominates  $W_j - A_1^j - \dots - A_{i-1}^j$ .

Since  $G$  is  $s$ -connected,  $|S_j| \geq s$ ,  $j = 1, 2$ , and there are  $s$  pairwise vertex disjoint  $S_1, S_2$ -paths  $P_1, \dots, P_s$ . We may assume that the only common vertex of  $P_i$  with  $S_j$  is  $p_{ij}$ . By Lemma 6(b), each  $p_{ij}$  has at least  $0.5t - 200s \log_2 t$  neighbors in  $W_j$ . Thus, we can choose  $2s$  distinct vertices  $q_{ij}$  such that  $q_{ij} \in W_j - \bigcup_{k=1}^s A_k^j$  and  $p_{ij}q_{ij} \in E(G)$ .

Define  $F_i = A_i^1 \cup A_i^2 \cup V(P_i) + q_{i1} + q_{i2}$ ,  $i = 1, \dots, s$ . Then for every  $i = 1, \dots, s$ ,

- (i)  $G[F_i]$  is connected;
- (ii)  $F_i$ -s are pairwise disjoint;
- (iii)  $F_i$  dominates  $\bigcup_{j=1}^2 W_j - F_1 - \dots - F_{i-1}$ .

Since by (2),

$$\left| \bigcup_{j=1}^2 W_j - F_1 - \dots - F_{i-1} \right| \geq 2(t - 400s \log_2 t) - 14.46s \log_2 1.1t - 2s > t,$$

$G$  has a  $K_{s,t}^*$ -minor, a contradiction.  $\square$

Comments. 1. Lemma 8 was reproved in [6] in a slightly stronger form.

2. The factor 1000 in (2) and maybe the factor 8 in front of  $s \log_2 s$  in Theorem 4 can be improved with more work, but Proposition 3 shows that the theorem will not hold if we replace both 1000 and 8 with  $1/18$ . Still, as Deryk Osthus observed, it could be that the statement holds for all  $s \leq t$  if we do not change 8.

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